

# General covariance, and supersymmetry without supersymmetry

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## Abstract

An unusual four-dimensional generally covariant and supersymmetric  $SU(2)$  gauge theory is described. The theory has local degrees of freedom, and is invariant under a local (left-handed) chiral supersymmetry, which is half the supersymmetry of supergravity. The Hamiltonian 3+1 decomposition of the theory reveals the remarkable feature that the local supersymmetry is a consequence of Yang-Mills symmetry, in a manner reminiscent of how general coordinate invariance in Chern-Simons theory is a consequence of Yang-Mills symmetry. It is possible to write down an infinite number of conserved currents, which strongly suggests that the theory is classically integrable. A possible scheme for non-perturbative quantization is outlined. This utilizes ideas that have been developed and applied recently to the problem of quantizing gravity.

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## I. INTRODUCTION

In any theory with local gauge symmetries, the symmetries of the action are normally manifest in the Hamiltonian theory as first class constraints on the phase space variables. Standard examples of this are the Gauss law constraint on the phase space of Yang-Mills theory, and the space and time reparametrization constraints, called respectively the spatial-diffeomorphism and Hamiltonian constraints, on the phase space of general relativity.

There are however exceptions to this rule. In topological field theories such as the  $SU(N)$   $BF$  theory [1], the action is invariant under both general coordinate transformations and Yang-Mills gauge transformations. However the Hamiltonian theory has *more* first class constraints than is indicated by these symmetries of the action. It is these extra constraints that are responsible for eliminating all local (or propagating) degrees of freedom, leaving only a finite number of global or topological degrees of freedom. For example, for  $SU(N)$   $BF$  theory in 3+1 dimensions, there are  $4(N^2 - 1)$  first class constraints per space point whereas the configuration space has  $3(N^2 - 1)$  degrees of freedom per space point.

Another well known example of this type is Chern-Simons theory, where for gauge group  $SU(N)$ , there are  $N^2 - 1$  first class constraints per point on  $2(N^2 - 1)$  phase space degrees of freedom per point; the usual counting after imposing gauge fixing conditions therefore gives no local degrees of freedom.

In general, in  $SU(N)$  topological field theories there are *more* first class constraints on the phase space than the  $(N^2 - 1) + 4$  that one might at first guess from the  $N^2 - 1$  Yang-Mills and four spacetime reparametrization invariances of the action. This happens in the  $B - F$  and Chern-Simons theories because the field equations imply that the Yang-Mills connections is flat.

There is an example of a four-dimensional theory [2] in which, unlike topological theories, there are *less* first class constraints on the phase space than is evident from an inspection of the symmetries of the action. The action for this theory, which resembles that for general relativity in the first order Palatini form, is

$$S = \int e^{AB} \wedge e_{BC} \wedge F_C^A, \quad (1.1)$$

where  $e_\mu^{AB}$  is a 1-form and  $F(A) = dA + A \wedge A$  is the curvature of the Yang-Mills connection  $A_\mu^{AB}$ . The Yang Mills gauge group is  $SU(2)$  ( $A, B, \dots = 1, 2$  are  $SU(2)$  2-spinor indices), and  $\mu\nu, \dots$  are spacetime indices. This would be an action for general relativity if the gauge group is taken to be  $SL(2, C)$  instead of  $SU(2)$ . Then instead of the dreibeins  $e_\mu^{AB} = e_\mu^{(AB)}$ , there would be vierbeins  $e_\mu^{AB'}$  in the action (where as usual the  $A'$  etc. are complex conjugate spinor indices), with no other change, that is, the connection would still be  $SU(2)$  valued [3,4]. Such an action leads to the Ashtekar Hamiltonian formulation of general relativity [5].

This theory (1.1) is not metric independent; the spacetime metric is  $g_{\mu\nu} = e_\mu^i e_\nu^j \delta_{ij}$ . However, because  $e_\mu^i$  is a dreibein, this metric is degenerate with signature  $(0 + ++)$ . The theory has genuine local degrees of freedom; it is a true field theory rather than a topological field theory with only a finite number of physical degrees of freedom. One way to see this is to count the number of first class phase space constraints: there is the  $SU(2)$  Gauss law constraint and, as shown in [2], *only* three other constraints which are the spatial diffeomorphism constraints. It turns out that the Hamiltonian constraint corresponding to time reparametrizations is not an additional constraint, but rather vanishes identically. The configuration space variable is the spatial component of the  $SU(2)$  connection, so there are  $9 - 3 - 3 = 3$  degrees of freedom per spatial point. Thus this theory has one more local degree of freedom per point than general relativity.

If we wish to quantize this theory, we might first consider perturbation theory. However, there is no expansion of the action (1.1) that isolates a quadratic ‘free theory’ term and a ‘non-linear’ interaction term, and so one cannot construct a perturbative quantum theory in the usual way. The same is true for general relativity in the first order covariant formulation, and it has been suggested [6] that this is the essential reason that general relativity is non-renormalizable as a perturbative quantum theory. Hence one would conclude, for the same reason, that the theory (1.1) is also non-renormalizable. However it has been shown that the quantum theory exists non-perturbatively [7]: There is a Hilbert space with well defined operators acting in the space. Thus the action (1.1) leads to a completely integrable quantum field theory!

As far as classically integrability is concerned, it is possible to write down an infinite number of constants of the motion: Since the Hamiltonian constraint vanishes identically, the spatial integral of any density constructed from the 3-metric is a constant of the motion; every invariant of three-geometry ( $\equiv$  three-metrics modulo diffeomorphisms) is a constant of the motion. However no complete Poisson commuting set is known that would explicitly prove Liouville integrability.

We turn now to generally covariant field theories with local supersymmetry. The only examples of such theories are supergravity and certain topological field theories with supergroup Yang-Mills invariance. In this paper we describe an unusual theory of this type which is neither of these, and which has a number of interesting properties.

The theory has local degrees of freedom, and two unusual features. The first feature is that *the local supersymmetry of the action is a consequence of its local Yang-Mills symmetry*. This is reminiscent of how general covariance in Chern-Simons theory is a consequence of Yang-Mills symmetry [6]. The second is that the Hamiltonian constraint corresponding to the time reparametrization symmetry is identically zero. This feature is just like that for the action (1.1) above. For this reason the theory we discuss in this paper may be viewed

as a supersymmetric version of (1.1).

The motivation for studying such a theory, apart from curiosity, is at least threefold: (i) Because of the unusual features described above, it is possible to write down explicitly an infinite number of constants of the motion, which strongly suggests that the theory is integrable. If a proof of integrability can be given, say in the sense of presenting the equations of motion in a Lax form, it would provide the first example of an integrable supersymmetric and generally covariant field theory in four dimensions with local degrees of freedom, (ii) the methods used in establishing the existence of the quantum theory of the action (1.1) may, with some modification, be used here as well to present a complete quantization, and (iii) the fact that the spacetime metric is degenerate and the Hamiltonian constraint is identically zero means that the theory is in a sense only three-dimensional: The initial data given on a three-dimensional Cauchy surface does not change. Although the action is a four-dimensional one, it is in this sense ‘already dimensionally reduced’. This suggests looking for higher dimensional generalizations and string type actions with the similar features.

The outline of this paper is as follows: In the next section we give the action and discuss its symmetries and field equations. Section III contains the Hamiltonian version of the theory, with an explanation of how the supersymmetry is realized at the canonical level. Section IV is a discussion of classical observables, and Section V contains a brief description of quantization. This is followed by a concluding section.

## II. THE MODEL

The theory we consider in this paper is given by the action

$$S = \int_M [ e^{AB} \wedge e_{BC} \wedge F_A^C + \alpha e^{AB} \wedge \psi_B \wedge D\psi_A ], \quad (2.1)$$

where  $e_\mu^{AB}$  is a one-form (bosonic) dreibein field,  $\psi_\mu^A$  is an anticommuting (fermionic) chiral spin 3/2 field, and  $A_{\mu A}^B$  is the (bosonic)  $SU(2)$  gauge field. The indices  $A, B, \dots$  are (chiral) two-spinor indices, and the covariant derivative and curvature are defined as usual by

$$\begin{aligned} D\lambda_A &= d\lambda_A + A_A^B \wedge \lambda_B, \\ F_A^B &= dA_A^B + A_A^C \wedge A_C^B. \end{aligned} \quad (2.2)$$

The conventions for raising and lowering spinor indices with the antisymmetric spinor  $\epsilon^{AB}$ , and its inverse  $\epsilon_{AB}$ , are  $\lambda^A = \epsilon^{AB} \lambda_B$  and  $\lambda_A = \lambda^B \epsilon_{BA}$ , where the  $\epsilon$ ’s satisfy  $\epsilon^{AC} \epsilon_{BC} = \delta_B^A$ . The fields  $\psi_\mu^A$ , being anticommuting, satisfy the conditions

$$\psi^A \wedge \psi^B = \psi^B \wedge \psi^A. \quad (2.3)$$

From this it follows that

$$\psi^A \wedge \psi_A = 0. \quad (2.4)$$

This action, being the integral of a four-form, is manifestly invariant under spacetime diffeomorphisms. It is also invariant under local  $SU(2)$  gauge transformations. The action has an additional local boson-fermion symmetry when the coupling constant  $\alpha = 1$ . As we now show, this is an on shell local supersymmetry. Consider the local transformations

$$\delta_\lambda e_{AB} = \psi_{(A} \lambda_{B)} \quad \delta_\lambda \psi_A = -D\lambda_A, \quad (2.5)$$

where  $\lambda_A(x)$  is an anticommuting (spacetime dependent) parameter. These satisfy the (anti-)commutation rules

$$[\delta_\lambda, \delta_\rho] \psi^A = 0, \quad \{\delta_\lambda, \delta_\rho\} \psi^A = 0, \quad (2.6)$$

$$\begin{aligned} [\delta_\lambda, \delta_\rho] e^{AB} &= \rho^{(A} \delta_\lambda \psi^{B)} + \lambda^{(A} \delta_\rho \psi^{B)} = D(\rho^{(A} \lambda^{B)}), \\ \{\delta_\lambda, \delta_\rho\} e^{AB} &= \rho^{(A} \delta_\lambda \psi^{B)} - \lambda^{(A} \delta_\rho \psi^{B)} = \rho^{(A} D\lambda^{B)} + \lambda^{(A} D\rho^{B)}. \end{aligned} \quad (2.7)$$

These relations are the left-handed component of the supersymmetry transformations in supergravity. (See for example [9].)

Under the transformations (2.5), the change in the lagrangian is

$$\begin{aligned} \delta_\lambda L &= 2(1 - \alpha) \psi^{(A} \lambda^{B)} \wedge e_{BC} \wedge F_A{}^C + \alpha \lambda_A \wedge \left( \frac{1}{2} \psi^B \wedge \psi^A - D e^{AB} \right) \wedge D \psi_B \\ &+ \text{surface terms.} \end{aligned} \quad (2.8)$$

For the parameter value  $\alpha = 1$  the action is invariant under the local supersymmetry transformations (2.5): The first term in the variation above vanishes, and the second term becomes proportional to the equation of motion for  $A_A{}^B$ , which, for  $\alpha = 1$ , is

$$D(e^{BA} \wedge e^C{}_B) + e^{B(A} \wedge \psi^{C)} \wedge \psi_B = 0. \quad (2.9)$$

By expansion of the symmetrization on the r.h.s., this last equation implies

$$D e^{AB} = \frac{1}{2} \psi^A \wedge \psi^B. \quad (2.10)$$

Using this, the second term in (2.8) vanishes. This establishes that (2.1) is invariant under the local supersymmetry generated by (2.5), modulo the equation of motion (2.9). (It is also the case in supergravity that one of the chiral supersymmetries is on shell in exactly this way [9]. See below.)

It is straightforward to make the action invariant under local supersymmetry without the use of an equation of motion. This is accomplished by extending the supersymmetry

transformations (2.5) to act also on the gauge field  $A_A^B$ . The necessary transformation on  $A_A^B$  may be deduced from the variation of the action. It is non-linear (and unconventional), and given by

$$\delta_\lambda A_{AC} \wedge e_B^C = -\frac{1}{2} \lambda_{(A} D\psi_{B)}. \quad (2.11)$$

The lagrangian for our model is similar to the following first order (complex) lagrangian for supergravity [9], which is made from only the self-dual part of the spin connection:

$$S_{SUGRA} = \int_M [ i e^{AA'} \wedge e_{A'B} \wedge F_A^B - e^{AA'} \wedge \bar{\psi}_{A'} \wedge D\psi_A ]. \quad (2.12)$$

Here the fields  $e_\mu^{AA'}$  are vierbiens rather than dreibeins, and both the unprimed  $\psi_\mu^A$  and their complex conjugate primed  $\psi_\mu^{A'}$  spinor fields are present. This supergravity action is separately invariant under left- and right-handed supersymmetry transformations, with invariance under the right-handed part being modulo the equations of motion for  $A_\mu^{AB}$ . These transformations are, respectively,

$$\delta_\lambda e_{AA'} = -i \bar{\psi}_{A'} \lambda_B, \quad \delta_\lambda \psi_A = 2D\lambda_A, \quad \delta_\lambda \bar{\psi}_{A'} = 0, \quad (2.13)$$

and

$$\delta_{\bar{\lambda}} e_{AA'} = -i \psi_A \bar{\lambda}_{A'}, \quad \delta_{\bar{\lambda}} \bar{\psi}_{A'} = 2D\bar{\lambda}_{A'}, \quad \delta_{\bar{\lambda}} \psi_A = 0, \quad (2.14)$$

together with  $\delta_\lambda A^{AB} = \delta_{\bar{\lambda}} A^{AB} = 0$ . There is therefore an additional ‘right-handed’ supersymmetry transformation in supergravity which is absent in our model. It is this additional transformation which, when anti-commuted with the left-handed transformation, gives the usual spacetime translation generator in supergravity. Also, the number of bosonic and fermionic fields in supergravity,  $e^{AA'}$  and the pair  $(\psi^A, \bar{\psi}^{A'})$ , are equal in number, whereas in our model there is a mismatch with twelve  $e^{AB}$  and only eight  $\psi^A$ . While this is unusual, the transformations (2.5) are manifestly still a local boson-fermion symmetry. For comparison, we note that the supersymmetry in the heterotic string is similar - the supersymmetry generator is a chiral spinor [10]. We note also that this is not the so called ‘ $\kappa$ -supersymmetry’ [10], which has the property that there is no associated conserved Noether charge. The variation of the action under the chiral supersymmetry above does lead to a non-trivial Noether charge. This is also reflected in the Hamiltonian theory below, in the fact that there is a first class constraint associated with the supersymmetry.

A further difference from supergravity is that there is a special spacetime direction in our theory which arises essentially because there is a dreibein field in four dimensions. This direction is given by the vector density

$$\tilde{u}^\alpha = \epsilon^{\alpha\beta\gamma\delta} e_\beta^{AB} e_{\gamma B}^C e_{\delta AC}, \quad (2.15)$$

where  $\epsilon^{\alpha\beta\gamma\delta}$  is the metric independent Levi-Civita tensor density, and is orthogonal to  $e_\mu^{AB}$ :  $\tilde{u}^\mu e_{\mu AB} = 0$ . There is a corresponding special 2-spinor density given by

$$\tilde{\phi}^A = \tilde{u}^\mu \psi_\mu^A. \quad (2.16)$$

The spacetime metric is  $g_{\mu\nu} = e_\mu^{AB} e_{\nu BA}$  and is degenerate with signature  $(0 + + +)$ , where the degeneracy direction is  $\tilde{u}^\mu$ . This situation is identical to the non-supersymmetric theory given in [2].

The other equations of motion following from (2.1), obtained by varying  $e_\mu^{AB}$  and  $\psi_a^A$  (with  $\alpha = 1$ ), are respectively

$$2 e_{C(B} \wedge F_{A)}^C + \psi_{(B} \wedge D\psi_{A)} = 0, \quad (2.17)$$

$$\psi_B \wedge D\epsilon^{AB} = 0. \quad (2.18)$$

We notice that this last equation (2.18) is identically satisfied when the equation of motion for  $A_a^{AB}$  (2.10) holds, together with the spinor identity (2.4). As we will see in the next section, it is due to this fact that the supersymmetry turns out to be a consequence of Yang-Mills symmetry.

### III. HAMILTONIAN THEORY

We now consider the Hamiltonian formulation of the theory considered in the last section. Assuming that the 4-manifold  $M = \Sigma \times R$ , where  $\Sigma$  is a spatial 3-manifold and  $R$  is ‘time’, it is straightforward to rewrite the action (2.1) in a  $3 + 1$  form. This gives

$$S = \int_R dt \int_\Sigma d^3x [ E^{cA}_B \dot{A}_{aA}^B + \Pi^{aB} \dot{\psi}_{aB} + e_0^{AB} \epsilon^{abc} ( 2e_{aBC} F_{bcA}^C - \psi_{aB} D_b \psi_{cA} ) \\ + A_{0A}^C ( D_a E^{aA}_C + \Pi^{aA} \psi_{aC} ) + \psi_{0A} ( \epsilon^{abc} e_a^{AB} D_b \psi_{cB} - D_a \Pi^{aA} ) ], \quad (3.1)$$

where  $a, b, \dots$  are there dimensional tensor indices,  $e_0^{AB}, A_{0A}^B$  and  $\psi_{0A}$  are the time components of the various fields, and

$$\epsilon^{abc} := \epsilon^{0abc} \quad (3.2)$$

is the 3-dimensional Levi-Civita tensor density. The variables

$$E^{aA}_B := 2\epsilon^{abc} e_b^{AC} e_{cCB} \quad \text{and} \quad \Pi^{aA} := -\epsilon^{abc} e_b^{AB} \psi_{cB}, \quad (3.3)$$

are the canonical momenta conjugate to the configuration variables  $A_{aA}^B$  and  $\psi_{aA}$  respectively. Thus the phase space is parametrized by the boson variables  $(A_{aA}^B, E^{aA}_B)$  and the fermionic variables  $(\psi_{aA}, \Pi^{aA})$ . It is instructive to write down the fundamental equal time Poisson brackets for the fermionic variables:

$$\{\epsilon^{abc}e_b^{AB}\psi_{cB}(x), \psi_{dD}(y)\} = \delta_d^a\delta_D^A\delta^3(x, y). \quad (3.4)$$

Since  $e_a^{AB}$  is effectively the variable canonically conjugate to  $A_a^{AB}$ , three combinations of the spinor and space components of  $\psi_{aA}$  are canonically conjugate to three other of its components. Therefore the fermionic part of the configuration space is effectively coordinatized by three functions and not six.<sup>1</sup> (An analagous feature appears in Hamiltonian Chern-Simons theory, where the components of the Yang-Mills connection are canonically conjugate to one another.) To indicate that only half the functions in  $\psi_a^A$  are configuration coordinates, we set

$$\psi_a^A = \phi_a\delta_1^A, \quad \Pi^{aA} = \pi^a\delta_1^A \quad (3.5)$$

for all that follows, where  $\phi_a$  and  $\pi^a$  are anticommuting variables.

Since the triads  $e_a^{AB}$  are assumed to be non-degenerate, the density

$$e := \frac{1}{3!}\epsilon^{abc}e_{aB}^A e_{bC}^B e_{cA}^C \neq 0, \quad (3.6)$$

and the canonical variables  $E^{aA}_B$  are dual to the triads  $e_{aB}^A$ :

$$E^{aA}_B = e e^{aA}_B, \quad (3.7)$$

with

$$e^{aA}_B e_{aC}^B = \delta_C^A \quad \text{and} \quad e^{aA}_B e_{bA}^B = \delta_b^a. \quad (3.8)$$

The variation of the 3+1 action (3.1) with respect to the non-dynamical time components of the fields  $e_0^{AB}$ ,  $A_{0A}^B$  and  $\psi_{0A}$ , leads to the three constraints

$$G_B^A := D_a E^{aA}_B + \Pi^{aA}\psi_{aB} = 0, \quad (3.9)$$

$$\epsilon^{abc} [ 2e_{aC}^{(B} F_{bc}^{A)C} - \psi_a^{(B} D_b \psi_c^{A)} ] = 0, \quad (3.10)$$

$$\epsilon^{abc} e_a^{AB} D_b \psi_{cB} - D_a \Pi^{aA} = 0, \quad (3.11)$$

where the symmetrization on the spinor indices in the second equation is due to  $e_a^{AB} = e_a^{BA}$ .

As expected in a generally covariant theory, the Hamiltonian is a linear combination of constraints, with  $e_0^{AB}$ ,  $A_{0A}^B$  and  $\psi_{0A}$  as the lagrange multipliers. On general grounds we expect that there will be constraints on the phase space associated with every local symmetry of the action. Therefore there should be four constraints associated with the 3-space and time reparametrization invariance, two constraints associated with the ‘right-handed’ local

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<sup>1</sup>I would like to thank Ted Jacobson for clarifying this point.



supersymmetry transformations (2.5), and finally three more constraints associated with the  $SU(2)$  Yang-Mills invariance. To see if this expectation is borne out, we must first rewrite the constraints as functions of only the canonical variables, and exhibit their Poisson algebra.

The first constraint (3.9) is the usual Gauss law associated with the  $SU(2)$  Yang-Mills invariance, and it is already a function of only the phase space variables.

The second constraint may be rewritten as a function of the phase space variables by multiplying it by  $\epsilon^{dab}e_d^{AB}$  and then using third constraint. The result is

$$E^{aA}_C F_{abA}^C - \Pi^{aA} D_b \psi_{aA} + \partial_a (\Pi^{aA} \psi_{bA}) = 0. \quad (3.12)$$

By adding to this a term proportional to the Gauss law constraint, specifically  $A_{aA}^B \times G_B^A$ , it becomes the spatial diffeomorphism constraint

$$C_b := E^{aA}_C \partial_a A_{bA}^C - \partial_a (E^{aA}_C A_{bA}^C) - \Pi^{aA} \partial_b \psi_{aA} + \partial_a (\Pi^{aA} \psi_{bA}) = 0. \quad (3.13)$$

Finally, using the equation

$$\epsilon^{abc} e_a^{AB} = \frac{E^{[bA}_C E^{c]CB}}{\sqrt{E}}, \quad (3.14)$$

where  $E = \epsilon_{abc} E_B^{aA} E_C^{bB} E_A^{cC}$ , the third constraint may also be rewritten as a function of the phase space variables:

$$S^A := \frac{E^{[bA}_C E^{c]CB}}{\sqrt{E}} D_b \psi_{cB} - D_a \Pi^{aA} = 0. \quad (3.15)$$

As expected, this constraint generates supersymmetry transformations on the phase space variables: With

$$S(\lambda) := \int_{\Sigma} \lambda_A S^A, \quad (3.16)$$

we have, for example,

$$\delta_{\lambda} \psi_A := \{\psi_A, S(\lambda)\} = -D\lambda_A, \quad (3.17)$$

which is one of the supersymmetry transformations (2.5) above.

We now point out a rather surprising feature of this supersymmetry constraint, namely, that *the supersymmetry constraint is identically satisfied as a consequence of the  $SU(2)$  Gauss law*. To show this, we first note from (3.11) that

$$S^A := \epsilon^{abc} D_b (e_a^{AB} \psi_{cB}) - \epsilon^{abc} \psi_{cB} D_b e_a^{AB} - D_a \Pi^{aA} = -\epsilon^{abc} \psi_{cB} D_b e_a^{AB} = 0, \quad (3.18)$$

where we have used the definition of the momentum conjugate to  $\psi_{aA}$  (3.3). This last form of the supersymmetry constraint (3.18), which is just the spatial projection of the field equation (2.18), and the Gauss law (3.9) may be succinctly written as<sup>2</sup>

$$S^A = \psi_B \wedge g^{AB} = 0, \quad (3.19)$$

$$G^{AB} = 2e_B^{(A} \wedge g^{B)C} = 0, \quad (3.20)$$

where the 2-form  $g^{AB}$  is defined by

$$g^{AB} := De^{AB} - \frac{1}{2}\psi^A \wedge \psi^B. \quad (3.21)$$

Now, it can be shown directly by expansion from (3.20) that

$$g_{[ab]}^{AB} = e^{cA}_C G_{[abc]}^{CB} - \frac{\epsilon^{def}}{\sqrt{E}} e_{[a}^{AB} G_{b]ef}^{CD} e_{dCD}. \quad (3.22)$$

Therefore we have

$$G^{AB} = 0 \iff g^{AB} = 0, \quad (3.23)$$

a result which was asserted above in equation (2.10).

It is now clear that the third constraint, may be written as a function of the Gauss law by using (3.22) in (3.19). This gives the explicit formula

$$S^A = \psi_B \wedge \left( e^{cA}_C G_{[abc]}^{CB} - \frac{\epsilon^{def}}{\sqrt{E}} e_{[a}^{AB} G_{b]ef}^{CD} e_{dCD} \right) dx^a \wedge dx^b. \quad (3.24)$$

Thus we have shown that only the first two of the constraints that follow from the action are independent. The third constraint, which as we saw above generates supersymmetry transformations, turns out to be identically satisfied when the  $SU(2)$  Gauss law holds. This is just the reflection in the Hamiltonian theory of the fact that the equation of motion (2.18) is identically satisfied as a consequence of (2.10). Thus there are effectively only the spatial diffeomorphism and Gauss law constraints in the Hamiltonian theory.

Putting all the above observations together, the final Hamiltonian form of the action, written entirely in terms of the phase space variables and appropriate lagrange multiplier functions, becomes

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<sup>2</sup>We are now writing these constraints as 3-forms rather than as scalar densities. The difference is multiplication by the Levi-Civita density  $\epsilon^{abc}$ .

$$S = \int_R dt \int_\Sigma d^3x [ E^{cA}_B \dot{A}_{aA}^B - N^a C_a - \mu_A^C G_C^A - \lambda_A^C G_C^A ], \quad (3.25)$$

where

$$N^a := e^{aA}_B e_{0A}^B, \quad (3.26)$$

$$\mu_A^C := A_{0A}^C - N^a A_{aA}^C, \quad (3.27)$$

$$\lambda_A^C := \psi_{0B} (\psi_{aA} e^{aBC} - \frac{\epsilon^{abc}}{\sqrt{E}} e_a^{DB} e_{bA}^C \psi_{cD}). \quad (3.28)$$

The Hamiltonian equations of motion are obtained by varying this action with respect to the canonical variables  $E_B^{cA}$ ,  $A_{aA}^B$ , and the variables  $N^a$ ,  $\mu_A^C$ , and  $\lambda_A^C$ . Varying these gives the two constraints

$$C_a = 0 = G_A^B. \quad (3.29)$$

Varying the canonical variables gives the Hamiltonian equations of motion

$$\dot{E}_{aA}^{aA} = \{E_{aA}^{aA}, H\}, \quad \dot{A}_{aA}^B = \{A_{aA}^B, H\}, \quad (3.30)$$

where the Hamiltonian is

$$H = \int_\Sigma d^3x [ N^a C_a + \mu_A^B G_B^A + \lambda_A^B G_B^A ]. \quad (3.31)$$

Since  $C_a$  is the generator of spatial diffeomorphisms, and  $G_B^A$  the generator of Gauss rotations, the evolution equations are simply

$$\dot{A}_{aA}^B = \mathcal{L}_N A_{aA}^B + D_a \mu_A^B + D_a \lambda_A^B, \quad (3.32)$$

and

$$\dot{E}_{aA}^{aA} = \mathcal{L}_N E_{aA}^{aA} + \mu_A^C E_C^{aA} + \lambda_A^C E_C^{aA} \quad (3.33)$$

where  $\mathcal{L}$  denotes the Lie derivative. We therefore see that ‘evolution’ in this theory amounts to spatial diffeomorphisms, and a pair of Gauss rotations of the canonical variables. In particular, *the supersymmetry of the covariant action manifests itself in the Hamiltonian theory only as a second Gauss transformation.*

Our model does for local supersymmetry what Chern-Simons theory does for general coordinate invariance: Local supersymmetry is a consequence of local Yang-Mills symmetry just as general coordinate invariance in Chern-Simons theory is a consequence of local Yang-Mills symmetry. In both these cases, the generators of the respective symmetries are functions of the Gauss law generator.

While we have shown this directly at the Hamiltonian level, it may be shown at the covariant level as well by appropriately constructing the Gauss transformation function out

of the physical fields and an arbitrary grassmann variable. A fundamental difference from Chern-Simons theory is, of course, that our model has local degrees of freedom - there are six independent configuration degrees of freedom per point.

Apart from the supersymmetry being a consequence of Yang-Mills symmetry, the theory has another unusual feature: The Hamiltonian constraint, which is the generator of time reparametrization invariance, does not appear in the Hamiltonian action (3.25). This feature is already present in the absence of grassmann fields, and has been discussed in detail in Ref. [2].

A comparison of the Hamiltonian theory of this model with Hamiltonian supergravity, (for example the one derived from the action (2.12)), provides a further understanding of the supersymmetry and spacetime reparametrization constraints. On the phase space of supergravity there are two first class constraints associated with the left and right-handed supersymmetry transformations (2.14-2.13). The Poisson bracket of these two constraints yields, in one guise or another [11–13], the generator of spacetime reparametrizations - the Hamiltonian and spatial-diffeomorphism constraints. In spinorial variables, these last two constraints arise together as one constraint with a pair of left and right-handed spinor indices, in the form  $\mathcal{H}_{AA'} = 0$ . It is in this way that the supersymmetry generators close to give the spacetime reparametrization generators. By contrast, in our model there is only the left-handed supersymmetry and the time reparametrization constraint vanishes identically. The supersymmetry constraint turns out to be proportional to the Gauss constraint, indirectly giving closure of the supersymmetry constraint algebra. (This is one way of seeing that the supersymmetry transformations close.)

#### IV. OBSERVABLES

There are different points of view about what is an observable in a generally covariant theory. An observable in any theory with first class constraints is defined to be a phase space function(al) which Poisson commutes with all the first class constraints. This basically gives the gauge invariant phase space variables, whose dynamics may then be studied using the Hamiltonian of the theory. In a generally covariant theory, the dynamics is generated by a constraint because the theory is invariant under time reparametrizations. If we apply the above prescription for finding the observables, we would be seeking constants of motion, which in fact do not evolve. This has led to suggestions that observables in a generally covariant theory should Poisson commute with only the ‘kinematical’ first class constraints.

In our model, the Hamiltonian constraint vanishes identically so there is no issue about how to define observables: These are phase space functionals that Poisson commute with the Gauss and spatial-diffeomorphism constraints. Such functionals are also constants of

motion. It is easy to write down an infinite number of them - the spatial integral of any Gauss law invariant scalar density of weight one. These are naturally divided into three classes: Those involving only the gravitational variables, only the fermionic variables, or both. The ‘electric’ 3-metric made from the dreibein  $E^{aAB}$ , and the ‘magnetic’ 3-metric made from  $H^{aAB} = \epsilon^{abc} F_{bc}^{AB}$  may be used to construct the scalar curvatures  $R(E)$  and  $R(H)$ . (We are assuming invertibility of the respective 3-metrics.) Examples of each of the three types of observables are

$$\int_{\Sigma} d^3x (\det E) R(H), \quad (4.1)$$

$$\int_{\Sigma} d^3x \Pi^{Aa} \psi_{aA}, \quad (4.2)$$

and

$$\int_{\Sigma} d^3x \Phi(R(H), R(E)) \Pi^{Aa} \psi_{aA}, \quad (4.3)$$

where  $\Phi$  is an arbitrary function of the curvature scalars of the electric and magnetic metrics. There are clearly an infinite number of such examples.

There are also diffeomorphism invariant loop observables which are constructed by defining loops using the phase space variables, rather than introducing loops as auxilliary variables [14]. This is done by defining loops as the intersections of two 2-dimensional surfaces, where the surfaces themselves are defined as the level surfaces of scalar fields made from the phase space variables. One example is the loop  $\gamma(c_1, c_2)$  defined by setting

$$f(R(E), R(H)) = c_1 \quad g(R(E), R(H)) = c_2, \quad (4.4)$$

where  $f, g$  are arbitrary functions of the Ricci scalars of the electric and magnetic metrics. This loop is constructed using only the gravitational variables. Then, the Wilson loop

$$W[\gamma, E, A] = \text{TrPexp} \int_{\gamma(c_1, c_2)} dx^a A_a, \quad (4.5)$$

based on such a ‘matter loop’, is a constant of the motion. One can do similar things using the spinor variables  $\Pi^{aA}$  and  $\psi_a^A$ , and also define loop observables with insertions along the loop [14]. This gives spatial-diffeomorphism invariant versions of the Rovelli-Smolín loop variables for canonical gravity [15].

The standard criteria for integrability is an algorithm for generating an infinite number of Poisson commuting constants of the motion. For 2-dimensional theories, or theories which can be effectively written as 2-dimensional theories such as self-dual gravity, such an algorithm is provided by the Lax, or zero-curvature form, of the evolution equations. In the absence of a similar procedure here, the fact that we can write down an infinite number of constants of motion is only suggestive that this model is integrable.

## V. QUANTIZATION

In this section we briefly consider the quantization of the model described in the preceding sections. As we have already pointed out, the action does not lead to a clear separation of ‘free theory’ and ‘perturbation’ terms. Therefore there does not seem to be a way to construct a perturbative quantum field theory starting from the action (2.1). We therefore consider non-perturbative Dirac quantization. A possible choice of representation for the quantum theory is the connection representation, where the wavefunctionals are  $\Phi[A_{aA}^B, \psi_a^A]$ . There are two sets of Dirac quantization conditions.

$$G_B^A |\Phi\rangle = \left[ D_a \frac{\delta}{\delta A_{aA}^B} + \psi_a^A \frac{\delta}{\delta \psi_a^B} \right] \Phi[A_{aA}^B, \psi_a^A] = 0, \quad (5.1)$$

$$\begin{aligned} C_b |\Phi\rangle = & \left[ (\partial_a A_{bA}^B) \frac{\delta}{\delta A_{aA}^B} - \partial_a (A_{bA}^B \frac{\delta}{\delta A_{aA}^B}) \right. \\ & \left. - (\partial_b \psi_a^A) \frac{\delta}{\delta \psi_a^A} + \partial_a (\psi_b^A \frac{\delta}{\delta \psi_a^A}) \right] \Phi[A_{aA}^B, \psi_a^A] = 0. \end{aligned} \quad (5.2)$$

The first condition states that  $|\Phi\rangle$  is invariant under  $SU(2)$  gauge transformations *and* under the left-handed supersymmetry transformations. This is because, as explained in the preceding sections, the latter are a consequence of the former. The second states that  $|\Phi\rangle$  is invariant under spatial diffeomorphisms. This is a remarkably simple prescription for obtaining the quantum states. Furthermore, since all the quantum constraints are linear in the momenta, there is no operator ordering ambiguity, and the quantum constraint algebra closes in the same way as the classical Poisson algebra.

It is a straightforward exercise to write down any number of quantum states: These are those observables of the last section which are functionals of only the  $A_{aA}^B$  and  $\psi_a^A$ . A class of purely bosonic states are traces of the Wilson loops, for loops  $\gamma(c_1, c_2)$  defined by  $f(R(H)) = c_1$ ,  $g(R(H)) = c_2$ . There are no purely fermionic states because it is not possible to form a non-zero scalar density using only  $\psi_a^A$ . Mixed states may be constructed by using the spinor density

$$\tilde{\chi}_A = \epsilon^{abc} H_{aAB} H_b^{BC} \psi_{cC}, \quad (5.3)$$

to form the scalar density  $\chi = \sqrt{\epsilon_{AB} \tilde{\chi}^A \tilde{\chi}^B}$ . A class of mixed states is then

$$\phi = \int_{\Sigma} d^3x \chi f(R(H), \sqrt{\chi}), \quad (5.4)$$

where  $f$  is an arbitrary function of its arguments. It is straightforward to produce many other examples. What is lacking is a systematic way of constructing a Hilbert space, which in turn is connected with producing a closed infinite dimensional algebra of physical observables.

A systematic approach for constructing a quantum theory may be to find a suitable generalization of the methods of Ashtekar et. al. in [7], which were developed for application

to diffeomorphism invariant theories of connections, such as general relativity in the Ashtekar formulation [5]. These methods have already been used to show that the theory given in [2], (which is the bosonic action (1.1) above), is an integrable quantum field theory. To see the form that such a generalization might take, we first give a brief outline of the main steps used in this approach, which is applicable to theories where the only configuration space variable is a connection:

1. The Gauss law invariant states are functions  $\Psi(\bar{A})$  on the space of generalized connections modulo gauge transformations, where the generalization is a suitable enlargement of the classical configuration space - the space of *smooth* connections modulo gauge transformations. Generalized connections may be distributional as well as smooth.
2. There is an innerproduct, and an orthonormal ‘spin network’ basis on the space of generalized connections which is labelled by closed graphs [7,16,17]. Associated with each edge of the graph is a matrix which is the holonomy of  $\bar{A}$  in a fixed representation (‘color’). Associated with each vertex of the graph is an ‘intertwiner’ matrix, which ties up all the matrix indices on the edges meeting at that vertex. This gives gauge invariance. Any finite number of edges can meet at a vertex. (These spin networks are a generalization of Penrose’s spin networks [18]).
3. The diffeomorphism constraint is implemented on this space, using this basis, via ‘exponentiation’. A unitary operator representing finite diffeomorphisms can be defined. Using this operator, diffeomorphism invariant states are constructed by ‘integrating over the group’. This gives an infinite class of quantum states. There is a natural inner product on the space of diffeomorphism invariant states obtained in this way.

To apply a similar procedure to the present model requires incorporating the Rarita-Schwinger spinor fields  $\psi_a^A$ , which are now a part of the configuration space.

The first step, which produces Gauss law invariant spin network states, may be generalized by allowing insertions of the  $\psi_a^A$  at the vertices, along with the intertwiner matrices. A class of open graphs also give gauge invariant states. These are the graphs whose open ends are the ends of edges in the fundamental representation of  $SU(2)$ . Such ends can be plugged with a  $\psi_a^A$  to give gauge invariance. The simplest such graph is one edge in the fundamental representation, with a  $\psi_a^A$  at each end.

It appears, at least at first sight, that the second step goes through as well. The orthonormality of the spin network states comes from ‘integration over the connection’, which is really a group integration over the holonomies associated with the edges of graphs. Therefore insertions of  $\psi_a^A$ ’s on the vertices do not effect these integrations. What they do effect

are the degeneracies - the number of graphs with a fixed number of edges and representations, but with the indices at the vertices tied up in different ways. A major difference, of course, is that the spin network states now also carry space indices (since the  $\psi_a^A$ 's do), which may make implementation of the third step more difficult.

## VI. DISCUSSION

We have described a model in which supersymmetry appears in a slightly different light - as a consequence of Yang-Mills symmetry. For this reason, perhaps it should not be called supersymmetry at all, at least at the Hamiltonian level. However, at the level of the action there is manifestly a chiral supersymmetry. A concomitant feature of this constraint structure is that the Hamiltonian constraint vanishes identically, which is to be contrasted with supergravity, where the left- and right-handed supersymmetry constraints close on the spacetime reparametrization constraints. It is the fact that the time reparametrization constraint vanishes identically that allows us to write down an infinite number of constants of motion, and also gives a possible interpretation of the theory as one that is already dimensionally reduced. From the point of view of the Hamiltonian evolution equations, the only change in the initial data under evolution is that due to spatial-diffeomorphisms and Yang-Mills gauge transformations.

It would be worthwhile to see if the quantization scheme suggested above can be carried to completion. If so, it would provide the first example of an integrable four-dimensional supersymmetric quantum field theory, as well as a concrete way to introduce matter into the methods developed for pure connection theories.

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